

Relativistic diffusion of particles with a continuous mass spectrum

Z. Haba

Institute of Theoretical Physics, University of Wrocław,
50-204 Wrocław, Plac Maxa Born 9, Poland
email:zhab@ift.uni.wroc.pl

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Abstract

We discuss general positivity conditions necessary for a definition of a relativistic diffusion on the phase space. We show that Lorentz covariant random vector fields on the forward cone $p^2 \geq 0$ lead to a definition of a generator of Lorentz covariant diffusions. We discuss in more detail diffusions arising from particle dynamics in a random electromagnetic field approximating the quantum field at finite temperature. We develop statistical mechanics of a gas of diffusing particles. We discuss viscosity of such a gas in an expansion of the energy momentum tensor in gradients of the fluid velocity.

1 Introduction

An approximation of the dynamics of a non-relativistic particle in an environment of other particles or random fields by a diffusion process has a long history of success [1][2]. A direct generalization of the definition to the diffusion in configuration Minkowski space does not work because the Minkowski space does not have a non-negative scalar product [3][4] (see the reviews in [5][6]). One could approach the problem of finding a realization of the diffusion by means of random dynamics [2][7]. Kramers diffusion in the phase space can be obtained by means of a random perturbation of Hamiltonian dynamics [7][8][9]. A relativistic generalization of the Kramers diffusion is uniquely determined by the requirement that the diffusion process should stay on the mass-shell. Then, its generator must be the Laplace-Beltrami operator on the mass-shell [10][11]. In this paper we relax the mass-shell condition admitting a continuous mass spectrum (see some earlier papers on diffusions in four-dimensional momentum space [12][13]). In fact, only stable free particles have a definite mass in relativistic

quantum field theory. Composite particles and resonances would have a certain mass distribution. When we give up the mass-shell condition then the manifold of possible relativistic diffusions is increasing substantially (a transport theory of quantum particles with a continuous mass spectrum has been initiated in [14], see also [15]). Any Lorentz covariant positive definite matrix defines a diffusion. However, if the diffusion matrix is to depend only on the momentum p^μ then its arbitrariness is restricted to a scalar function of p^2 . Such diffusions have no equilibrium (they are analogs of the Brownian motion). On general physical grounds the equilibrium depends on the frame of reference [16][17][18]. We can describe the frame by a time-like unit vector w^μ . If the diffusion matrix can depend on both w and p then the set of diffusions is much larger. We determine the diffusions resulting from the requirement that they should have the covariant (depending on p and w) Jüttner equilibrium distribution [19].

The plan of the paper is the following. In the next section we discuss a positivity condition resulting from the assumption that the diffusion is defined by a probability measure and is relativistic invariant. An unexpected relation of the diffusion matrix to the energy momentum tensor is pointed out. This relation is suggesting methods for a construction of relativistic diffusions. In this section we introduce still another method of finding a positive definite and Lorentz covariant diffusion matrix: the random dynamics. The square of a Lorentz covariant vector field defines a positive definite diffusion matrix. If Φ is the probability distribution of the diffusion process then it defines a current N^μ and the energy momentum tensor of a stream of diffusing particles. In sec.3 we investigate whether these (vector and tensor) currents can be conserved in analogy to the non-relativistic Brownian motion (in the momentum space). We determine the diffusion matrix and the drift of such conservative diffusions. In sec.4 we briefly discuss a relation of the expectation value of the square of a random vector field to the random dynamics (Kubo's diffusion approximation [20][21]). In sec.5 we consider a particle motion in a random scalar and electromagnetic fields. We calculate an expectation value of the square of the vector field defining the random dynamics. We show that the diffusion generator determined by the random dynamics depends only on the field two-point correlation function at coinciding points. The correlation functions at finite temperature depend on the reference frame described by the unit time-like vector w^μ . Using the vector w^μ we can define a drift by means of an expectation value of the electromagnetic field $F_{\mu\nu}$. We have introduced such a drift in our earlier papers [22][23] as a friction which brings the diffusion to the Jüttner equilibrium on the basis of the detailed balance condition (a generalization of the Ornstein-Uhlenbeck process). In sec.5 we show that the diffusion process with a continuous mass spectrum can be decomposed into the processes on the mass-shell introduced first by Schay [10] and Dudley[11]. In sec.7 we introduce some notions of (non-equilibrium) statistical physics (energy, entropy, free energy) for a stream of diffusing particles and investigate their time evolution and a relation between them. In secs.7 and 8 we initiate a hydrodynamic description of a stream of

relativistic diffusing particles in analogy to the well-known description resulting from the Boltzmann equation [24][25][26]. We show that the diffusion equation leads to hydrodynamic equations similar to the ones of the Boltzmann fluid. As a consequence we can apply the standard methods of an expansion in gradients [25][26] for an approximation of these equations by a viscous (Navier-Stokes) flow.

2 Some general positivity conditions on the diffusion tensor

We begin with a general consideration of requirements imposed on the relativistic diffusion. Explicitly Lorentz invariant evolution can be formulated in terms of the proper time. Diffusion can be considered as classical dynamics in a random field. In such an approach the proper time arises in the description of the relativistic diffusion. Assume that there is a transition function $P_\tau(x, p; x', p')$ in the proper time τ . Then, the diffusion generator (restricted to the momentum variables) is

$$\mathcal{A} = d^{\mu\nu} \partial_\mu \partial_\nu + b^\mu \partial_\mu = \partial_\mu (d^{\mu\nu} \partial_\nu + b^\mu) - \partial_\mu b^\mu - (\partial_\nu d^{\mu\nu}) \partial_\mu \quad (1)$$

(we denote the derivatives over p^μ by ∂_μ and over x^μ by ∂_μ^x). In eq.(1) we use two alternative ways to write the generator of the diffusion which will be discussed later on. From the definition of the diffusion matrix

$$\begin{aligned} d^{\mu\nu} &= \lim_{\tau \rightarrow 0} \langle \tau^{-1} (p^\mu(\tau) - p^\mu(0)) (p^\nu(\tau) - p^\nu(0)) \rangle \\ &= \lim_{\tau \rightarrow 0} \tau^{-1} \int dp' P_\tau(x, p; x', p') (p'^\mu - p^\mu) (p'^\nu - p^\nu), \end{aligned} \quad (2)$$

where $\langle \dots \rangle$ denotes a probabilistic expectation value. It follows

$$\begin{aligned} a_\mu a_\nu d^{\mu\nu} &= \lim_{\tau \rightarrow 0} \langle a_\mu a_\nu \tau^{-1} (p^\mu(\tau) - p^\mu(0)) (p^\nu(\tau) - p^\nu(0)) \rangle \\ &= \lim_{\tau \rightarrow 0} \tau^{-1} \langle \left(a(p(\tau) - p(0)) \right)^2 \rangle \geq 0. \end{aligned} \quad (3)$$

If the process is Lorentz invariant then the generator (1) must be built from Lorentz tensors. It is easy to see that a homogeneous ($d^{\mu\nu}$ independent of p) process does not exist if $d^{\mu\nu}$ is to be built solely from the Minkowski metric $\eta^{\mu\nu} = (1, -1, -1, -1)$ because $\eta^{\mu\nu}$ is not positive definite. Eqs.(1)-(3) hold true if $\tau = x_0$ (the coordinate time) and $x \rightarrow \mathbf{x}$ (only the space coordinates diffuse) but the Lorentz invariance is not explicit. A formulation of the Lorentz invariance is more complicated in this case.

We may extend the search for relativistic diffusions to generally covariant theories. We note that from equations of general relativity the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \theta_{\mu\nu}, \quad (4)$$

where $R_{\mu\nu}$ is the Ricci tensor and $g_{\mu\nu}$ is a pseudoRiemannian metric, satisfies the positivity requirement

$$a_\mu a_\nu G^{\mu\nu} \geq 0,$$

if the energy momentum $\theta_{\mu\nu}$ is positive definite. Then, both the Einstein tensor and $\theta_{\mu\nu}$ could be chosen as the diffusion tensors. As an example of $\theta_{\mu\nu}$ we could take the one of an ideal fluid (discussed further in this paper) or the energy form of a map $\phi: M \rightarrow \mathcal{M}$ between Riemannian manifolds [27]

$$\theta_{\mu\nu} = g_{ab}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b, \quad (5)$$

where M is the Minkowski space-time and \mathcal{M} is the Riemannian manifold equipped with the metric g_{ab} .

A general positive definite matrix $T^{\mu\nu}$ has the representation [28]

$$T^{\mu\nu} = \int dk \tilde{G}(k) k^\mu k^\nu, \quad (6)$$

where $\tilde{G}(k) \geq 0$. In the model (5) we obtain

$$\tilde{G}\delta(k+k') = \langle g_{ab}(\phi) \tilde{\phi}^a(k) \tilde{\phi}^b(k') \rangle, \quad (7)$$

if the expectation value $\langle \dots \rangle$ is translation invariant ($\tilde{\phi}$ denotes the Fourier transform of ϕ).

If \tilde{G} depends only on k and is Lorentz invariant then the argument that a positive definite $d^{\mu\nu}$ does not exist (as discussed after eq.(3)) still applies. We cannot define $T^{\mu\nu}$ by means of eq.(6) using a Lorentz invariant $\tilde{G}(k)$ function of k because for such $\tilde{G}(k)$ the integral (6) would be divergent. We shall assume that $\tilde{G}(k, w)$ is a Lorentz invariant function of k and an additional vector $w \in V_+$ (without the loss of generality we may assume $w^2 = 1$). In such a case the tensor $T^{\mu\nu}$ must be of the form

$$T^{\mu\nu}(x) = \alpha(x) w^\mu(x) w^\nu(x) - \omega(x) (\eta^{\mu\nu} - w^\mu(x) w^\nu(x)), \quad (8)$$

where $\eta^{\mu\nu}$ is the Minkowski metric, α and ω are some Lorentz invariant non-negative functions. We can express them by T

$$\alpha = \int dk \tilde{G}(wk)^2 = w^\mu T_{\mu\nu} w^\nu, \quad (9)$$

$$\omega = \frac{1}{3} \int dk \tilde{G}((kw)^2 - k^2) = \frac{1}{3} (w^\mu T_{\mu\nu} w^\nu - T_\mu^\mu). \quad (10)$$

If the support of \tilde{G} is in V_+ then $\alpha \geq 3\omega$ as

$$\alpha = 3\omega + \int dk \tilde{G}k^2. \quad (11)$$

By means of the tensor $T^{\mu\nu}$ we can define some new diffusion tensors , e.g.,

$$d_{\mu\sigma} = (-\eta_{\mu\sigma}T_{\nu\rho} + \eta_{\mu\rho}T_{\nu\sigma} - \eta_{\nu\rho}T_{\sigma\mu} + \eta_{\nu\sigma}T_{\mu\rho})p^\nu p^\rho. \quad (12)$$

If \tilde{G} is non-negative and is vanishing for $k^2 < 0$ then eq.(12) defines an admissible diffusion tensor (satisfying the inequality (3)), as we show in sec.5.

There is another way to find a positive diffusion matrix. We can find a diffusion generator assuming random dynamics. Let $A(s)$ be a random flow (the first order differential operator) depending on the proper time s . We consider

$$Y_\tau = \int_0^\tau ds A(s) \quad (13)$$

Then, we define

$$Y^2 = \lim_{\tau \rightarrow 0} Y_\tau^2 \tau^{-2} = \lim_{\tau \rightarrow 0} \tau^{-2} \frac{1}{2} \int_0^\tau \int_0^\tau ds ds' (A(s)A(s') + A(s')A(s)). \quad (14)$$

If $Y = \alpha^\mu \partial_\mu$ then (Y^+ denotes an adjoint of Y in $L^2(dp)$)

$$\langle Y^2 \rangle = -\langle Y^+ Y \rangle + \langle \alpha^\mu \partial_\mu \alpha^\nu \rangle \partial_\nu.$$

The first operator on the rhs is negative definite (because $Y^+ Y$ is non-negative). It is of the form $-d^{\mu\nu} \partial_\mu \partial_\nu + b^\mu \partial_\mu$. Then, it follows from the theory of partial differential operators of the second order ([29], sec.12) that the matrix $d^{\mu\nu}$ satisfies the positivity condition (3) (because the operator $Y^+ Y$ must be elliptic). The operator (14) in general is degenerate, i.e., reduced to a lower dimensional (as a rule one dimensional) second order differential operator. In general, it is only covariant under Lorentz transformations. If the random field is Lorentz covariant and we take a mean value of the square in eq.(14) then we obtain a non-degenerate invariant negative definite second order differential operator on the phase space. This is a good candidate for a generator of the relativistic diffusion. The randomness may come from a classical approximation of quantum fluctuations. In the next section we discuss some additional conditions imposed on the diffusion equation resulting from its probabilistic interpretation. In the subsequent sections we show that the methods (6) and (14) are related to each other. Moreover, the diffusion generated by Y^2 of eq.(14) approximates random dynamics.

3 The relativistic diffusion with a continuous mass spectrum

We define the diffusive evolution of observables (functions of position and momentum) by the differential equation in the proper time

$$\partial_\tau \phi = p^\mu \partial_\mu^x \phi + \mathcal{A} \phi. \quad (15)$$

An evolution of the probability distribution Φ in the proper time τ is determined by the duality

$$\Phi_\tau(\phi) = \int dp dx \sigma(p, x) \Phi(p, x) \phi_\tau(p, x), \quad (16)$$

where σ is a Lorentz invariant non-negative function (distribution). The dynamics of the probability distribution Φ in the coordinate (laboratory) time is derived from the requirement that Φ does not depend on τ . This condition gives the transport equation

$$p^\mu \partial_\mu^x \Phi = \mathcal{A}^+ \Phi = d^{\mu\nu} \partial_\mu \partial_\nu \Phi + B^\mu \partial_\mu \Phi + Q\Phi, \quad (17)$$

where \mathcal{A}^+ is the adjoint of \mathcal{A} (1) in $L^2(dp\sigma)$. The functions B and Q can be determined from the definition of the adjoint operator and eq.(1). For the probabilistic interpretation we need a conservation of the current

$$N^\mu = \int dp \sigma p^\mu \Phi. \quad (18)$$

A Lorentz invariant distribution σ in quantum theory of resonances is of the form

$$\sigma(p^2) = \int dm^2 \delta(p^2 - m^2) s(m^2) \theta(p_0), \quad (19)$$

where m has the meaning of a mass and θ denotes the Heaviside step function. If σ is independent of x then the requirement $\partial_\mu^x N^\mu = 0$ leads to the equation

$$\partial_\alpha \partial_\rho (d^{\alpha\rho} \sigma) - \partial_\alpha (B^\alpha \sigma) + Q\sigma = 0. \quad (20)$$

(under the assumptions that σd is vanishing at $p = 0$ and $p = \infty$, so that the boundaries do not contribute to the integration by parts formula).

We assume first that the coefficients in \mathcal{A}^+ depend only on p . From eq.(20) we obtain $p^\alpha d_{\alpha\rho} = 0$ (because $\partial_\alpha \sigma = \sigma' 2p_\alpha$ and from eq.(20) $d^{\alpha\rho} \partial_\rho \sigma$ must vanish if eq.(20) is to be satisfied for any $\sigma(p^2)$). Hence,

$$d^{\mu\nu} = P^{\mu\rho} C_{\rho\alpha} P^{\alpha\nu}. \quad (21)$$

where $P^{\mu\nu}$ is the projection operator

$$P^{\mu\nu} = \eta^{\mu\nu} - (p^2)^{-1} p^\mu p^\nu. \quad (22)$$

If the diffusion depends only on p then from the Lorentz invariance $C_{\mu\rho} = -\eta_{\mu\rho} p^2 \gamma$. From eq.(21)

$$d^{\mu\nu} = -\gamma p^2 P^{\mu\nu} \quad (23)$$

and

$$b^\mu = \lambda p^\mu, \quad (24)$$

where λ and $\gamma \geq 0$ are functions of p^2 . Let us note that the matrix $d^{\mu\nu}$ (23) can be derived from eq.(6) when

$$\phi^\mu = (x^\mu - p^\mu p x) \sqrt{\gamma}$$

with $g_{ab} \rightarrow \eta_{\mu\nu}$. The matrix d of eq.(23) is positive definite if and only if the spectral condition arising from eq.(19) is satisfied.

The current N^μ is conserved if (from eq.(20))

$$Q\sigma = 4\lambda\sigma + 2(\lambda\sigma)'p^2 - 6(\gamma\sigma)'p^2 - 12\gamma\sigma. \quad (25)$$

We consider also the energy momentum tensor of a diffusing particle

$$\theta^{\mu\nu} = \int dp \sigma \Phi p^\mu p^\nu. \quad (26)$$

The energy momentum is conserved if

$$\partial_\alpha \partial_\rho (d^{\alpha\rho} p^\nu \sigma) - \partial_\alpha (p^\nu b^\alpha \sigma) + Q\sigma p^\nu = 0. \quad (27)$$

Explicitly, with the representation (23) and (24)

$$Q\sigma = 5\sigma\lambda - 18\gamma\sigma + 2(\lambda\sigma)'p^2 - 6(\gamma\sigma)'p^2.$$

We can see that both equations (25) and (27) (for the current conservation and the energy momentum conservation) are satisfied if

$$\gamma = \frac{1}{6}\lambda \equiv \kappa^2 a(p^2). \quad (28)$$

In such a case

$$\mathcal{A} = -\kappa^2 a(p^2) p^2 P^{\mu\nu} \partial_\mu \partial_\nu \quad (29)$$

(where $a(p^2) \geq 0$ is assumed to be a regular function at $p^2 = 0$ vanishing at infinity). The diffusion generated by the operator (29) is the analog of the non-relativistic (Kramers) Brownian motion which also gives a conservation of the current and the stress tensor ($T^{0j} = N^j, T^{lk}$) (the non-relativistic limit of eq.(17) with \mathcal{A} of eq.(29) coincides with the Kramers diffusion; the diffusion of Schay [10] and Dudley [11] has the same non-relativistic limit but does not preserve the energy momentum). If the current and energy momentum conservations are satisfied then the transport equation reads

$$p^\mu \partial_\mu^x \Phi = \mathcal{A}^+ \Phi = -\kappa^2 a(p^2) p^2 P^{\mu\nu} \partial_\mu \partial_\nu \Phi + 6\kappa^2 a(p^2) p^\mu \partial_\mu \Phi + Q\Phi, \quad (30)$$

where

$$Q = 12\kappa^2 + 6\kappa^2 \sigma^{-1} (\sigma a)' p^2. \quad (31)$$

The diffusion (30) has an immediate generalization to an arbitrary metric $g^{\mu\nu}$ on a pseudoRiemannian manifold M . For this purpose it is sufficient to replace $\eta^{\mu\nu}$ in the definition of $P^{\mu\nu}$ (22) by a general metric $g^{\mu\nu}$ and add the

geodesic correction $\Gamma_{\nu\sigma}^{\mu}p^{\nu}p^{\sigma}\partial_{\mu}\Phi$ (where Γ 's are the Christoffel symbols) on the rhs of eq.(30). In such a case the energy momentum tensor (26) of the diffusing particles can be put on the rhs of the Einstein equations (4). If the energy momentum is not conserved then a varying cosmological constant is needed in order to describe the Riemannian geometry of a diffusing matter [30].

The diffusion (30) has no equilibrium. An additional friction drift f^{μ} in eq.(30) is needed if the diffusion is to equilibrate. The equilibrium state depends on the frame of reference [17][18]. We denote the velocity of the frame by w^{μ} (where $w^{\mu}w_{\mu} = 1$). Then, the drift is to depend on p^{μ} and on w^{μ} . We consider a generalization of eq.(30) of the form

$$p^{\mu}\partial_{\mu}^x\Phi = \mathcal{A}_f^+\Phi = -\kappa^2a(p^2)p^2P^{\mu\nu}\partial_{\mu}\partial_{\nu}\Phi + 6\kappa^2a(p^2)p^{\mu}\partial_{\mu}\Phi + \sigma^{-1}\partial_{\mu}(f^{\mu}\Phi) + Q\Phi \quad (32)$$

We have inserted the drift f in eq.(32) in such a way that the current conservation $\partial_{\mu}^xN^{\mu} = 0$ is automatically preserved. With the drift f^{μ} the energy momentum conservation fails

$$\partial_{\mu}^x\theta^{\mu\nu} = -\int dp\Phi f^{\nu}. \quad (33)$$

Eq.(33) describes an exchange of energy with an environment (the heat bath). The equilibrium is determined by the requirement

$$\mathcal{A}_f^+\Phi_E = 0. \quad (34)$$

\mathcal{A}_f^+ can be written in the second form of eq.(1)

$$\mathcal{A}_f^+ = -\sigma^{-1}\partial_{\mu}(\kappa^2\sigma a(p^2)p^2P^{\mu\nu}\partial_{\nu} - 3\kappa^2\sigma a(p^2)p^{\mu} - f^{\mu}). \quad (35)$$

It follows from eq.(35) that if

$$f^{\mu} = \kappa^2\sigma ap^2P^{\mu\nu}\partial_{\nu}\ln\Phi_E - 3\kappa^2a\sigma p^{\mu}, \quad (36)$$

then eq.(34) for the equilibrium is satisfied. In particular, for the Jüttner distribution [19]

$$\Phi_E = \exp(-\beta wp - Kp^2) \quad (37)$$

(where K is an arbitrary non-negative constant and β has the meaning of the inverse temperature) we have

$$f^{\mu} = -\beta\kappa^2a\sigma p^2P^{\mu\nu}w_{\nu} - 3\kappa^2a\sigma p^{\mu}. \quad (38)$$

The divergence of the energy momentum is expressed again by the energy momentum and the conserved current (18) if $a = \sigma = 1$. Then

$$\partial_{\mu}^x\theta^{\mu\nu} = 3\kappa^2N^{\nu} - \beta\kappa^2(\theta^{\nu\mu}w_{\mu} - w^{\nu}\theta_{\mu}^{\mu}). \quad (39)$$

Eq.(39) remains true on the mass-shell [31]. We obtain eq.(36) in a model of the dynamics of a relativistic particle in a random electromagnetic field in sec.5. The non-relativistic limit of the diffusion (36) coincides with the Ornstein-Uhlenbeck process

$$\partial_t \Phi = -\mathbf{p} \nabla_{\mathbf{x}} \Phi + \partial_j (\partial_j + \beta(p_j - w_j)) \Phi,$$

which has a current conservation but no energy momentum conservation (because of friction).

Instead of adding a friction f^μ to eq.(30) we first could look for a general solution of the positivity requirement upon the diffusion matrix (21). We do not know a general solution of the problem. From a study of a particle motion in a random electromagnetic field in sec.5 we obtain

$$C^{\mu\nu} = -ap^2 \eta^{\mu\nu} + b(\eta^{\mu\nu} (pw)^2 + p^2 w^\mu w^\nu) \quad (40)$$

where a and b are non-negative constants. The friction term leading to the Jüttner equilibrium (37) will also be derived in sec.5 from a particle interaction with a random electromagnetic field.

4 Random dynamics

Explicitly Lorentz invariant relativistic dynamics can be expressed in the proper time τ [32]

$$\frac{dx^\mu}{d\tau} = p^\mu, \quad (41)$$

$$\frac{dp^\mu}{d\tau} = R^\mu(x, p). \quad (42)$$

A function W of observables evolves as

$$\partial_\tau W = p^\mu \partial_\mu^x W + R^\mu \partial_\mu W \equiv (X + Y)W, \quad (43)$$

where

$$X = p^\mu \partial_\mu^x. \quad (44)$$

The current (18) ($\Phi = W$) is conserved if

$$\int dp W \partial_\mu (R^\mu \sigma) = 0 \quad (45)$$

Let

$$Y(s) = \exp(-sX) Y \exp(sX) = R^\mu(x - ps, p) (\partial_\mu + s \partial_\mu^x), \quad (46)$$

where

$$Y = R^\mu \partial_\mu$$

Then, the solution of eq.(43) can be expressed in the form

$$W_\tau = \exp(\tau X) W_\tau^I, \quad (47)$$

where

$$\partial_s W_s^I = Y(s) W_s^I. \quad (48)$$

We assume that R^μ are random variables. In general, we have the cumulant expansion for the expectation value

$$\langle W_t \rangle = \exp(\tau X) \exp \left(\int_0^t ds \langle Y(s) \rangle + \frac{1}{4} \int_0^t ds \int_0^s ds' \langle (A(s)A(s') + A(s')A(s)) \rangle + \dots \right) W. \quad (49)$$

here

$$A(s) = Y(s) - \langle Y(s) \rangle$$

If $[Y(s), Y(s')] = 0$ and Y is a linear function of Gaussian variables then eq.(49) is exact (with no higher order terms). The approach of Kubo [20]-[21] approximates the random Liouville operator on the rhs of eq.(49) by an expectation value of its square.

The expansion of the dynamics (43) till the second order term reads

$$\begin{aligned} \langle W_\tau \rangle = & W + \tau p^\mu \partial_\mu^x W + \tau p^\mu \partial_\mu^x \int_0^\tau ds \langle Y(s) \rangle W \\ & + \int_0^\tau ds \langle Y(s) \rangle W + \frac{1}{2} \left(\int_0^\tau ds \langle Y(s) \rangle \right)^2 W + \frac{1}{2} \langle \left(\int_0^\tau ds A(s) \right)^2 \rangle W + \dots \end{aligned} \quad (50)$$

In the expansion (50) we find the τ^2 term of eq.(14). There is also the first order differential operator (of the first order in τ) which can be defined by

$$K_1 = \lim_{\tau \rightarrow 0} \int_0^\tau ds \langle Y(s) \rangle \tau^{-1}.$$

Then, there appears in eq.(50) a term which is of the first order in derivatives and the second order in time. It can be defined by means of the formula

$$K_2 = \lim_{\tau \rightarrow 0} \left(\int_0^\tau ds \langle Y(s) \rangle - K_1 \tau \right) \tau^{-2}. \quad (51)$$

$\langle Y^2 \rangle + K_2$ determines the diffusion generator (where Y^2 is defined in eq.(14)). The expansion (49) shows that the diffusion generated by $\langle Y^2 \rangle + K_2$ is related to random dynamics. In fact, Kubo shows [20] that the τ^2 behaviour in random dynamics at times short in comparison to the correlation time goes into the diffusive τ behaviour at times large in comparison to the correlation time. His argument is equivalent to the rigorous Markov approximation of ref.[9] which is using a time rescaling from a microscopic time to the macroscopic time. In the next section we calculate the expectation values in eq.(50) for a particle in a random electromagnetic field.

5 Motion in a random scalar and electromagnetic fields

The simplest example of R^μ in eq.(42) is defined by a scalar field ϕ

$$R_\mu = \partial_\mu^x \phi.$$

Then, Y^2 of eq.(14) has the form

$$\mathcal{A} = T^{\mu\nu} \partial_\mu \partial_\nu, \quad (52)$$

where $T^{\mu\nu}$ has been calculated in eqs.(6)-(8). We do not develop this diffusion any further because in the rest frame ($w = (1, 0, 0, 0)$ and $\alpha = 0$) it coincides with the Brownian motion. We can consider R^μ constructed from higher rank tensor fields and p^μ . In this paper we restrict ourselves to

$$R^\mu = \mathcal{R}^{\mu\nu} p_\nu, \quad (53)$$

where $\mathcal{R}_{\mu\nu}$ is the antisymmetric tensor of an electromagnetic field. The electromagnetic current is

$$\mathcal{J}_\nu = \partial^\mu \mathcal{R}_{\mu\nu} \quad (54)$$

We assume that

$$\langle \mathcal{J}^\mu \rangle = r w^\mu \quad (55)$$

with a certain constant r . We split

$$\mathcal{R}_{\mu\nu} = F_{\mu\nu}^{av} + F_{\mu\nu} \quad (56)$$

where F^{av} is the mean value of \mathcal{R} . $F_{\mu\nu}$ has zero mean value and is Poincare invariant. $r w_\nu$ has the meaning of a constant current. The current (54) results from a charge r moving with the velocity w^μ . We can obtain such a current in a finite temperature quantum field theory of interacting electromagnetic and complex scalar fields. The density matrix is

$$\exp(-\beta P^\nu w_\nu) \quad (57)$$

where P_μ is the four-momentum of the quantum fields and w^μ describes the moving frame. Then, calculating the expectation value (of the current (54) of the quantum scalar complex field ϕ interacting with a quantum electromagnetic field) we obtain

$$\begin{aligned} \langle \mathcal{J}_\mu \rangle_\beta &= i Tr \left(\exp(-\beta P^\nu w_\nu) (\bar{\phi} \partial_\mu^x \phi - \phi \partial_\mu^x \bar{\phi}) \right) \\ &= \int d\mathbf{k} k_0^{-1} k_\mu (\exp(\beta k w) - 1)^{-1} = r w_\mu \end{aligned} \quad (58)$$

(with a certain constant r). There is another way to see that an introduction of a current leads to a non-zero expectation value of $\partial_\sigma \mathcal{R}_{\mu\nu}$. If we add the term $\int A_0 d\mathbf{x}$ to the Hamiltonian of the quantum electromagnetic field or in a covariant and gauge invariant way the term $\int A_\mu w^\mu dx$ to the action (here A_μ is the electromagnetic vector potential) then $\langle \mathcal{R}_{\mu\nu, \sigma} \rangle \neq 0$.

We assume that an average $\langle \cdot \rangle$ over F is defined which preserves the Lorentz symmetry. This means that the two-point function defined by

$$\langle F_{\mu\nu}(x) F_{\sigma\rho}(x') \rangle = G_{\mu\nu; \sigma\rho}(x - x') \quad (59)$$

is a tensor. $G_{\mu\nu;\sigma\rho}$ is symmetric under the exchange of indices $(\mu\nu; x)$ and $(\sigma\rho; x')$ and antisymmetric under the exchange $\mu \rightarrow \nu$ and $\sigma \rightarrow \rho$. We impose the Bianchi identities

$$\mathcal{R}_{\mu\nu,\sigma} + \mathcal{R}_{\sigma\mu,\nu} + \mathcal{R}_{\nu\sigma,\mu} = 0 \quad (60)$$

on the tensor field \mathcal{R} . In terms of the two-point function

$$\partial_\alpha \epsilon^{\alpha\beta\mu\nu} G_{\mu\nu;\sigma\rho} = 0. \quad (61)$$

In Fourier transforms eq.(61) reads

$$\langle \bar{\tilde{F}}_{\mu\nu}(k) \tilde{F}_{\sigma\rho}(k') \rangle = \tilde{G}_{\mu\nu;\sigma\rho}(k) \delta(k - k'), \quad (62)$$

where $\tilde{G}_{\mu\nu;\sigma\rho}(k)$ is a tensor which must be constructed from the vectors k_μ, w_μ and the fundamental four-dimensional tensors $\eta_{\mu\rho}$ and $\epsilon_{\mu\nu\rho\sigma}$. Hence, in general we could have

$$\begin{aligned} \tilde{G}_{\mu\nu;\sigma\rho}(k) = & a_1(\eta_{\mu\sigma}k_\nu k_\rho - \eta_{\mu\rho}k_\nu k_\sigma + \eta_{\nu\rho}k_\mu k_\sigma - \eta_{\nu\sigma}k_\mu k_\rho) + a_0\epsilon_{\mu\nu\sigma\rho} \\ & + a_2(\eta_{\mu\sigma}w_\nu k_\rho - \eta_{\mu\rho}w_\nu k_\sigma + \eta_{\nu\rho}w_\mu k_\sigma - \eta_{\nu\sigma}w_\mu k_\rho) \\ & + a_3(\eta_{\mu\sigma}w_\nu w_\rho - \eta_{\mu\rho}w_\nu w_\sigma + \eta_{\nu\rho}w_\mu w_\sigma - \eta_{\nu\sigma}w_\mu w_\rho) \end{aligned} \quad (63)$$

However, the Bianchi identities (61) and the requirement of positivity of the probability measure in eq.(59) (see [33]) lead to $a_0 = a_2 = a_3 = 0$ (in the case of a quantum free electromagnetic field at finite temperature we obtain also the representation (63) where only $a_1 \neq 0$). We have

$$\begin{aligned} G_{\mu\nu;\sigma\rho}(x, x') = & \int dk \tilde{G}_{\mu\nu;\sigma\rho}(k) \exp(ik(x - x')) \\ = & \int dk \tilde{G}(k) \exp(ik(x - x')) (\eta_{\mu\sigma}k_\nu k_\rho - \eta_{\mu\rho}k_\nu k_\sigma + \eta_{\nu\rho}k_\mu k_\sigma - \eta_{\nu\sigma}k_\mu k_\rho) \end{aligned} \quad (64)$$

and

$$\langle F_{\mu\nu}(x) F_{\sigma\rho}(x') \rangle_\beta = -D_{\mu\nu;\sigma\rho} G(x - x'), \quad (65)$$

where

$$D_{\mu\nu;\sigma\rho} = -\eta_{\mu\sigma}\partial_\nu\partial_\rho + \eta_{\mu\rho}\partial_\nu\partial_\sigma - \eta_{\nu\rho}\partial_\sigma\partial_\mu + \eta_{\nu\sigma}\partial_\mu\partial_\rho \quad (66)$$

The two-point function is positive definite if and only if $\tilde{G}(k)$ in eq.(64) satisfies the condition

$$\tilde{G}(k) \geq 0 \quad (67)$$

and $\tilde{G}(k)=0$ if $k^2 < 0$ [33].

It follows from eq.(64) that

$$(D_{\mu\nu;\sigma\rho} G)(0) = \eta_{\mu\sigma} T_{\nu\rho} - \eta_{\mu\rho} T_{\nu\sigma} + \eta_{\nu\rho} T_{\sigma\mu} - \eta_{\nu\sigma} T_{\mu\rho}, \quad (68)$$

where $T_{\mu\nu}$ is defined in eq.(6).

It is instructive to see the relation between T and the energy-momentum tensor $\theta_{\mu\nu}$ for the electromagnetic field. We have

$$\theta_{\nu\rho} = \eta^{\mu\sigma} F_{\mu\nu} F_{\sigma\rho} - \frac{1}{4} \eta_{\nu\rho} F_{\alpha\sigma} F^{\alpha\sigma}. \quad (69)$$

Hence, the expectation value of the energy momentum tensor is expressed as

$$\langle \theta_{\mu\nu} \rangle = 2T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T^\rho_\rho \quad (70)$$

We can show that in eq.(51)

$$K_1 = \langle \mathcal{R}_{\mu\nu} \rangle p^\nu \partial^\mu \quad (71)$$

and that the rhs of eq.(52) can be expressed as

$$\left(\int_0^\tau ds \langle Y(s) \rangle - K_1 \tau \right) \tau^{-2} = \tau^{-2} \int_0^\tau ds \int_0^s ds' \partial_\sigma F_{\mu\nu}^{av} (x - (s - s')p) p^\sigma p^\nu \partial_\mu \quad (72)$$

Then, taking the limit $\tau \rightarrow 0$ in eq.(72) we obtain

$$K_2 = \langle \partial_\sigma \mathcal{R}_{\mu\nu} \rangle p^\sigma p^\nu \partial^\mu \quad (73)$$

We apply the covariance (65) to calculate the expectation value of the square of the Liouville operator. In this way we calculate the τ^2 term in the expansion (49). Then, ($A(s)$ is defined below eq.(49))

$$\begin{aligned} Y^2 &= \lim_{\tau \rightarrow 0} \frac{1}{\tau^2} \langle (\int_0^\tau A(s) ds)^2 \rangle = \lim_{\tau \rightarrow 0} \frac{1}{2\tau^2} \int_0^\tau ds \int_0^\tau ds' \langle A(s)A(s') + A(s')A(s) \rangle \\ &= (\eta^{\mu\sigma} T^{\nu\rho} - \eta^{\mu\rho} T^{\nu\sigma} + \eta^{\nu\rho} T^{\mu\sigma} - \eta^{\nu\sigma} T^{\mu\rho}) p_\nu \partial_\mu p_\rho \partial_\sigma \end{aligned} \quad (74)$$

If we apply the formulas (8) then we obtain

$$\begin{aligned} Y^2 &= -2\omega \partial_\mu P^{\mu\nu} p^2 \partial_\nu \\ &+ (\rho + \omega)((wp)^2 \partial_\mu \partial^\mu - 2wpw^\mu p^\rho \partial_\mu \partial_\rho + p^2 w^\mu w^\rho \partial_\mu \partial_\rho - w^2 p^\nu \partial_\nu - 2wpw^\nu \partial_\nu) \equiv \mathcal{A}_w \end{aligned} \quad (75)$$

The limit in eq.(74) depends only on $DG(0)$ of eqs.(65) and (68). It does not depend on p_μ . The dependence on p_μ of $D_{\mu\nu;\rho\sigma} G((s - s')p)$ in eq.(49) is of higher than the second order in τ (because after integration in eq.(49) over s and s' we obtain a factor τ^3).

The transport equation determined by the diffusion $Y^2 + K_2$ (eqs.(75) and (73)) reads (it is an extension of eqs.(21),(32) and (40))

$$\begin{aligned} p^\mu \partial_\mu^x \Phi &= \mathcal{A}_f^+ \Phi = \partial_\mu (D^\mu + f^\mu) \Phi \\ &\equiv \partial_\mu \left(-a P^{\mu\nu} p^2 \partial_\nu - r p^2 P^{\mu\nu} w_\nu + b((pw)^2 \partial^\mu - (wp)(w^\mu p^\rho + w^\rho p^\mu) \partial_\rho + p^2 w^\mu w^\rho \partial_\rho) \right) \Phi \end{aligned} \quad (76)$$

Here, $a = 2\omega$, $b = \rho + \omega$ and the drift (as discussed in sec.3) resulting from eq.(73) is

$$f^\mu = -r p^2 P^{\mu\nu} w_\nu. \quad (77)$$

r is an arbitrary real parameter which could be determined from the original model of random dynamics (as in eq.(58)).

6 Restriction to the mass-shell

First, we discuss the diffusion at zero temperature corresponding to $w = 0$ in eq. (76). The calculations of $\langle Y^2 \rangle$ in sec.5 lead to the following analog of the Brownian motion

$$\partial_\tau \Phi \equiv \mathcal{G}\Phi = p^\mu \partial_\mu^x \Phi + \kappa^2 \Delta_P \Phi \quad (78)$$

where κ^2 is a diffusion constant and

$$\Delta_P = -\partial_\mu P^{\mu\nu} p^2 \partial_\nu \quad (79)$$

and \mathcal{G} is an operator in the phase space of points (x, p) such that $p^2 \geq 0$ and $p_0 \geq 0$. The operator $-\Delta_P$ can be defined as a symmetric positive definite operator in $L^2(dp\sigma)$ (it is symmetric also in $L^2(dp)$)

$$(f_1, \Delta_P f_2) = D(P\partial f_1, p^2 P\partial f_2) = \int dp\sigma(p) p^2 P^{\mu\nu} \partial_\mu f_1 \partial_\nu f_2. \quad (80)$$

Let us consider the direct integral of Hilbert spaces

$$\mathcal{H} = \int dm^2 s(m^2) \mathcal{H}_m$$

where

$$(f_1, f_2)_m = \int d^4p \delta(p^2 - m^2) f_1(p) f_2(p) \quad (81)$$

Define the bilinear form

$$g(\partial f_1, \partial f_2)_m = \int d^4p \delta(p^2 - m^2) \gamma^{jk} \partial_j f_1 \partial_k f_2 \equiv (f_1, -\Delta_H^m f_2)_m \quad (82)$$

with

$$\gamma^{jk} = m^2 \delta^{jk} + p^j p^k \quad (83)$$

and

$$\Delta_H^m = \gamma^{-\frac{1}{2}} \partial_j \gamma^{jk} \gamma^{\frac{1}{2}} \partial_k. \quad (84)$$

The operator Δ_H^m is the Laplace-Beltrami operator on the hyperboloid $p^\mu p_\mu = m^2$. $\gamma = \det(\gamma_{jk})$ and $\gamma^{-1} = m^4 p_0^2$, $p_0 = \sqrt{m^2 + \mathbf{p}^2}$, $k = 1, 2, 3$. The operator Δ_H in Eq.(84) can be expressed in coordinates \mathbf{p} as

$$\Delta_H^m = (\delta^{jl} m^2 + p^j p^l) \frac{\partial}{\partial p^l} \frac{\partial}{\partial p^j} + 3p^l \frac{\partial}{\partial p^l} \quad (85)$$

We have

$$\int dm^2 s(m^2) g(\partial f_1, \partial f_2)_m = -D(P\partial f_1, p^2 P\partial f_2). \quad (86)$$

On the lhs of eq.(86) in each \mathcal{H}_m we can express the derivatives either by p_μ or by p_j treating p_0 as a function of p_j . The relation between the derivatives is

$$\tilde{\partial}_j = (\partial_j p_0) \partial_0 + \partial_j. \quad (87)$$

The functions f_j depend on a four-vector p . On the rhs of eq.(86) we have an explicitly Lorentz invariant formula expressed by Lorentz vectors and a measure which is a scalar. On the lhs the Lorentz invariance is not explicit. The lhs is explicitly positive definite whereas the positive definiteness of the rhs is not obvious.

Δ_H^m is the generator of the diffusion defined first by Schay[10] and Dudley[11]. The drift (77) can also be restricted to the mass-shell [22]-[23] (it determines the exponential speed of the decay to the equilibrium, see [34]).

7 Thermodynamics of diffusing particles

We are interested in this section in a thermodynamic description of a system of diffusing particles in models which have an equilibrium solution. We find the equilibrium solution in the model (76) from the requirement

$$(D^\mu - rp^2 P^{\mu\nu} w_\nu) \Phi_E = 0. \quad (88)$$

We obtain the Jüttner equilibrium (37) if

$$r = \beta(a - b) = \beta(\omega - \alpha). \quad (89)$$

Φ_E is also an equilibrium solution of the model (35)-(36) with $r = \kappa^2 \beta$.

From eq.(76) we obtain

$$\partial_\mu^x \Theta^{\mu\nu} = -(b - 3a)N^\nu + r\Theta_\mu^\mu w^\nu - r\Theta^{\mu\nu} w_\mu - 2bw^\nu w_\mu N^\mu. \quad (90)$$

In the model (35)-(36) we have $b = a = 0$ on the rhs of eq.(90).

We can introduce thermodynamic notions useful in a description of a stream of relativistic diffusing particles. We define the relative entropy current (Kulback-Leibler entropy; we follow definitions of our earlier paper [31] concerning the model on the mass-shell)

$$S_K^\mu = N^{-1} \int dp \sigma(p) p^\mu \Phi \ln(N^{-1} \Phi N_E \Phi_E^{-1}). \quad (91)$$

Here,

$$N = \int d\mathbf{x} N_0 \quad (92)$$

is the charge normalization constant (N_E in the case of the equilibrium). It follows that $\Omega = N^{-1} \sigma^{-1} p_0^{-1} \Phi$ has the meaning of the probability density (on the mass-shell $N^{-1} \Phi$ is the probability density). It can be shown that

$$\int d\mathbf{x} S_K^0 \geq 0 \quad (93)$$

and

$$\partial_\mu S_K^\mu = -N^{-1} \int dp \sigma(p) \Phi \alpha^{\mu\nu} \partial_\mu \ln R \partial_\nu \ln R, \quad (94)$$

where $R = \Phi \Phi_E^{-1}$. $\alpha^{\mu\nu}$ comes from the formula (76) where

$$D^\mu = \alpha^{\mu\nu} \partial_\nu. \quad (95)$$

From positive definiteness of $-\mathcal{A}_w$ the momentum dependent coefficients $\alpha^{\mu\nu}$ satisfy the positivity condition $a_\mu a_\nu \alpha^{\mu\nu} \geq 0$ (see our discussion after eq.(14)). It follows that $\int d\mathbf{x} S_K^0$ is a positive function decreasing monotonically to zero (it can be interpreted as the entropy of the system plus its heat bath). The entropy current of the particle system is

$$S^\mu = -N^{-1} \int dp \sigma(p) p^\mu \Phi \ln \Phi. \quad (96)$$

We have

$$\partial_\mu S^\mu = N^{-1} \int dp \sigma(p) \Phi \alpha^{\mu\nu} \partial_\mu \ln \Phi \partial_\nu \ln \Phi - 3r w_\mu N^\mu. \quad (97)$$

The entropy is defined as

$$S = \int d\mathbf{x} S^0. \quad (98)$$

Then

$$\partial_0 S = N^{-1} \int dp \sigma(p) \Phi \alpha^{\mu\nu} \partial_\mu \ln \Phi \partial_\nu \ln \Phi - 3r N^{-1} w_\mu \int d\mathbf{x} N^\mu. \quad (99)$$

We may choose the frame $w = (1, \mathbf{0})$. Then,

$$w^\mu \int d\mathbf{x} N_\mu = \int d\mathbf{x} N_0 = N \geq 0 \quad (100)$$

So, the first term on the rhs of eq.(99) is positive whereas the second is negative. We define the free energy

$$\mathcal{F} = \beta^{-1} N \int d\mathbf{x} S_K^0 - N \beta^{-1} \ln(N N_E^{-1}) \quad (101)$$

and the energy

$$\mathcal{W} = \int d\mathbf{x} \theta_{00}. \quad (102)$$

Then, the basic thermodynamic relation (at fixed temperature)

$$\beta^{-1} S = \mathcal{W} - \mathcal{F} \quad (103)$$

comes out as an identity. The time evolution of each term in eq.(103) is determined by eqs.(39),(91) and (99). From eq.(103) we can see that the decrease of $\partial_0 S$ in eq.(99) is caused by the decrease of \mathcal{W} in eq.(103).

8 Fluid velocity and the macroscopic energy momentum

On the basis of the diffusion theory we can develop a hydrodynamic description of a gas of diffusing particles. We define the density (a scalar)

$$\rho(\Phi) = \int dp \sigma(p) \Phi \quad (104)$$

and the mean momentum

$$v^\mu = \rho(\Phi)^{-1} N^\mu \quad (105)$$

We can write the energy-momentum tensor in the form

$$\theta^{\mu\nu} = \rho v^\mu v^\nu + \tau^{\mu\nu} \quad (106)$$

where

$$\tau^{\mu\nu} = \int dp \sigma(p) (p^\mu - v^\mu)(p^\nu - v^\nu) \Phi \quad (107)$$

Let

$$n^2 = v_\alpha v^\alpha \quad (108)$$

$$v^\mu = n u^\mu \quad (109)$$

Then, $u_\mu u^\mu = 1$. We define the projection operator

$$h^{\mu\nu}(u) = \eta^{\mu\nu} - u^\mu u^\nu \quad (110)$$

such that $u_\mu h^{\mu\nu} = 0$. In general, we can write

$$\tau^{\mu\nu} = -\Pi h^{\mu\nu} + \sigma^{\mu\nu} \quad (111)$$

(for a gas of free relativistic particles in an equilibrium we have $\sigma^{\mu\nu} = 0$). So, we write the energy momentum tensor in the form

$$\theta^{\mu\nu} = E u^\mu u^\nu - \Pi h^{\mu\nu} + \sigma^{\mu\nu} \quad (112)$$

where

$$E = \rho n^2 \quad (113)$$

We shall discuss in more detail the models (35)-(36) and (76) with $b = 0$ (in these models fluid equations have a simple interpretation). As discussed in [35][36][37] the relativistic fluid equations are projections of the conservation laws. Let us identify the divergence equations (39) with the equations resulting from the definition of the energy-momentum tensor (112) (calculating $h_{\alpha\nu} \partial_\mu \theta^{\mu\nu}$ in two ways)

$$\begin{aligned} & -(E + \Pi) u^\mu \partial_\mu u_\alpha + h_{\alpha\nu} \partial^\nu \Pi - h_{\alpha\nu} \partial_\mu \sigma^{\mu\nu} \\ & = \beta \kappa^2 h_{\alpha\nu} \left(-\Pi w^\nu + w_\mu \sigma^{\mu\nu} - w^\nu (E - 3\Pi + \sigma^\mu_\mu) \right). \end{aligned} \quad (114)$$

Let us note that the fluid equations do not depend on the N^ν -term in eq.(90). When there is no friction ($\beta = 0$) we obtain the same equations as the ones of Landau and Lifshitz [36]. The non-relativistic limit of eq.(114) gives (Euler or Navier-Stokes depending on $\sigma^{\mu\nu}$) fluid equations.

9 Viscosity of relativistic diffusing particles

The velocity w^μ in the transport equation (76) is an arbitrary auxiliary variable. In this section we allow w^μ to depend on x . We can interpret an equation with x -dependent w either as the diffusion with friction in a frame moving with the local velocity $w^\mu(x)$ or as the diffusion of a stream of particles in a fluid moving with the velocity $w^\mu(x)$ and playing the role of the heat bath. In the state

$$\Phi_E = \exp(\chi(x) - \beta(x)pw(x)) \quad (115)$$

the particle mean velocity is $u_E^\mu(x) = w^\mu(x)$. Φ_E satisfies the equation $\mathcal{A}_w^+ \Phi_E = 0$ but the lhs of eq.(76) is different from zero depending on the derivatives of the velocity w , temperature β^{-1} and the density $\exp(\chi(x))$. Then, $\Phi_E(\mathbf{x}, x_0)$ evolves from an initial time x_0 according to the diffusion equation (76). The initial Φ_E and its subsequent time evolution constitute a useful description of a state close to the local equilibrium. In this section we discuss the time evolution of Φ_E (115) and the fluid velocity in more detail ((115) is an analog of the local equilibrium of the non-relativistic Boltzmann equation discussed in an elementary way in [26] and in a more advanced form in [25]; for the relativistic case see [38][39] and [40]).

The mean momentum v^μ of the stream in the state Φ_E is

$$\rho_E v_E^\mu = \int dp \sigma(p) p^\mu \exp(-\beta p_\nu w^\nu(x)) = n_E \rho_E w^\mu(x), \quad (116)$$

where

$$\rho_E = \int dp \sigma(p) \exp(-\beta p_\nu w^\nu(x)).$$

n_E is the normalization of $v_E^\mu = n_E w^\mu$. We have

$$n_E \rho_E = \int dp \sigma(p) pw \exp(-\beta pw) = \int dm^2 s(m^2) n_m \rho_E^m \quad (117)$$

where n_m in

$$n_m \rho_E^m = \int dp \delta(p^2 - m^2) pw \exp(-\beta p_\nu w^\nu(x)) \quad (118)$$

is the value of n calculated on the mass-shell (see e.g. [39]).

In this section we consider the models (35)-(36) or (76) with $b = 0$ discussed in sec.7. We express the transport equation as the diffusion equation in an invariant time variable

$$\tau = x_\mu w^\mu \quad (119)$$

(different from τ in eqs.(15) and (41); the deterministic proper time satisfies $\tau = x_0$ in the particle rest frame whereas for the time (119) we obtain $\tau = x_0$ in the observer's rest frame). We may change the coordinates $(x_0, \mathbf{x}) \rightarrow (\tau, \mathbf{x})$ and $(p_0, \mathbf{p}) \rightarrow (q, \mathbf{p})$ where

$$q = w^\mu p_\mu \quad (120)$$

In such a case

$$p_\mu \partial_x^\mu = q \partial_\tau + \mathbf{p} \nabla_{\mathbf{x}} + p^\mu x_\nu \partial_\mu^x w^\nu \partial_\tau \equiv q \partial_\tau + \mathcal{D} \quad (121)$$

Let us consider

$$\Phi = \Gamma \Phi_E \quad (122)$$

Then,

$$p^\mu \partial_\mu^x \Gamma = \mathcal{A} \Gamma - (p^\mu \partial_\mu^x \ln \Phi_E) \Gamma \quad (123)$$

In new coordinates

$$q \partial_\tau \Gamma = \mathcal{A} \Gamma - \mathcal{D} \Gamma - (p^\mu \partial_\mu^x \ln \Phi_E) \Gamma \quad (124)$$

The solution of eq.(124) can be expressed by an exponential of $q^{-1}(\mathcal{A} - \mathcal{D})$. If in the lowest order we neglect the second order space-time derivatives and the squares of the first order derivatives of w in Γ then the solution of eq.(124) reads

$$\Gamma(x) = 1 - q^{-1} \int_0^\tau ds \exp((\tau - s)q^{-1}\mathcal{A}) p^\mu \partial_\mu^x \ln \Phi_E(s, \mathbf{x}) \quad (125)$$

Taking only the lowest eigenvalue $\nu > 0$ of $-q^{-1}\mathcal{A}$ in eq.(125) we obtain the relativistic relaxation time approximation from the formula

$$\int_0^\tau ds \exp(-\nu(\tau - s)) F(s) \simeq \nu^{-1} F(\tau) \quad (126)$$

true for large ν . Then, the lowest order relativistic relaxation time approximation reads

$$\Phi - \Phi_E = \nu^{-1} q^{-1} p^\mu \partial_\mu^x \Phi_E. \quad (127)$$

We obtain ν if we can calculate the time evolution (125). We are able to do this only in the high energy limit $q^{-1}p^2 \rightarrow 0$. Such a limit is equivalent to the limit $m \rightarrow 0$ in eq.(76). The time evolution (125) can be calculated exactly [41][31] in this limit and the relaxation time approximation is achieved with $\nu = \beta \kappa^2$. In general, in order to obtain a local in time fluid equation we need a local in time relativistic approximation to the solution (125). The result of such an approximation can be inferred from the relativistic invariance. The relativistic invariant (local in time) approximation to eq.(125) must be of the form

$$\begin{aligned} \Gamma(p, x) = & 1 + \partial_\mu^x w_\nu(x) (a_0 p^\mu p^\nu + a_1 w^\mu p^\nu) + a_2 p^\mu \partial_\mu \chi \\ & + a_3 p^\mu \partial_\mu \beta + a_4 w^\mu \partial_\mu \chi + a_5 w^\mu \partial_\mu \beta, \end{aligned} \quad (128)$$

where $a_j(\tau, pw, p^2)$ are scalars depending on the available Lorentz scalars τ, pw and p^2 . In the relaxation time approximation we have

$$a_0 = -(\nu pw)^{-1}$$

and $a_j = 0$ for $j > 0$.

Inserting this Γ in eqs.(18) and (105) we can obtain a formula for the current, from eq.(107) the formula for the energy momentum. We could derive the general tensor form of these expressions. Subsequently, from the equations of conservation of the current and the energy momentum we obtain differential equations for the velocity, temperature and the density. The calculations have been performed explicitly in the relaxation time approximation to the Boltzmann equation in the classic paper [42]. We would like to concentrate here on the appearance of the viscosity (which is of interest for high energy physics [43]) in the approximation (125). For this purpose it is sufficient if we restrict ourselves to $\beta = \text{const}$ and $\chi = \text{const}$. In such a case

$$\rho(\Phi)v_\mu = n_E \rho_E w_\mu + c_1 \rho_E w^\nu \partial_\nu^x w_\mu(x) + c_2 \rho_E w_\mu \partial_\nu^x w^\nu(x) \quad (129)$$

where c_1 and c_2 can be calculated from eqs.(18) and (105) as integrals over p depending on the functions a_0 and a_1 . In the case of the relaxation time approximation the constants c_1 and c_2 are determined by the integrals

$$s^{\mu_1 \dots \mu_n} = \int dm^2 s(m^2) \int dp \delta(p^2 - m^2) (pw)^{-1} \Phi_E p^{\mu_1} \dots p^{\mu_n} \quad (130)$$

calculated in [42]. The tensor on the rhs of (130) is expressed by w^μ and $\eta^{\mu\nu}$. The energy momentum tensor has the form

$$\begin{aligned} \theta^{\mu\nu} = & \theta_E^{\mu\nu} + \int d\sigma p^\mu p^\nu (\Gamma - 1) \Phi_E = (E + \Pi_E) w^\mu(x) w^\nu(x) - \eta^{\mu\nu} \Pi_E \\ & + \delta_0 (\partial_x^\mu w^\nu(x) + \partial_x^\nu w^\mu(x)) + \delta_4 \eta^{\mu\nu} \partial_\alpha^x w^\alpha - \delta_3 w^\mu(x) w^\nu(x) \partial_\alpha^x w^\alpha(x) \\ & - \delta_1 (w^\mu(x) w^\alpha(x) \partial_\alpha^x w^\nu(x) + w^\nu(x) w^\alpha(x) \partial_\alpha^x w^\mu(x)), \end{aligned} \quad (131)$$

where E and Π_E are the energy and pressure of the Jüttner gas (they are constants if χ and β are constants in eq.(115)). δ_j can be calculated if the functions a_0 and a_1 are known (in the relaxation time approximation we determine δ_j from integrals of the form (130)). We can express u by w from eq.(129)

$$u^\mu = w^\mu + c_1 n_E^{-1} w^\nu \partial_\nu^x w^\mu. \quad (132)$$

We can see that till the terms first order in gradients we have

$$\partial_\alpha^x u^\mu = \partial_\alpha^x w^\mu \quad (133)$$

and

$$w^\mu = u^\mu - c_1 n_E^{-1} u^\nu \partial_\nu^x u^\mu. \quad (134)$$

Now, we can replace w in $\theta^{\mu\nu}$ by u and its derivatives. We have

$$\theta^{\mu\nu} = (E + \Pi_E)u^\mu u^\nu - \eta^{\mu\nu}\Pi_E + \sigma_E^{\mu\nu} \quad (135)$$

where

$$\sigma_E^{\mu\nu} = -\sigma_1(u^\nu u^\sigma \partial_\sigma^x u^\mu + u^\mu u^\sigma \partial_\sigma^x u^\nu) + \delta_0(\partial_x^\mu u^\nu + \partial_x^\nu u^\mu) - \delta_3 u^\mu u^\nu \partial_\alpha^x u^\alpha + \delta_4 \eta^{\mu\nu} \partial_\alpha^x u^\alpha. \quad (136)$$

Here

$$\sigma_1 = \delta_1 + (E + \Pi_E)c_1 n_E^{-1} \quad (137)$$

The current conservation and eq.(133) give (when χ and β are kept constant in eq.(115))

$$\partial_\mu^x u^\mu = 0. \quad (138)$$

Eq.(114) rewritten in terms of $\sigma_E^{\mu\nu}$ (136) with w^μ (134) expressed by u^μ reads

$$\epsilon u^\mu \partial_\mu u_\alpha = -h_{\alpha\nu}(u)(\delta_0 \partial_x^\mu h_{\mu\sigma} \partial_x^\sigma u^\nu - (\sigma_1 - \delta_0)(u^\rho \partial_\rho^x)^2 u^\nu), \quad (139)$$

where

$$\epsilon = E + \Pi_E + \beta \kappa^2 (-3\Pi_E c_1 n_E^{-1} + \delta_0 - \delta_1). \quad (140)$$

In eq.(139) we have omitted the squares of the gradients of u^μ . This is the relativistic Navier-Stokes equation with the shear viscosity δ_0 . There is no gradient of the pressure and no bulk viscosity in eq.(139) because in the state (115) (with constant χ and β) in our approximation the fluid is incompressible and has constant pressure.

We can see that the friction changes the value of $E + \Pi_E$. Apart from this minor change we obtain the same hydrodynamics which we could have derived from the diffusion theory of sec.3 with $f^\mu = 0$ (preserving the conservation of the energy momentum; in the equation for energy balance the friction would appear). In our definition of u^μ as proportional to the current we have chosen the Eckart convention [35]. This means that in general the energy momentum tensor describes a certain heat flow. We would need $\sigma_1 = \delta_0$ to eliminate the heat flow. We do not expect such an equality to be satisfied for the same reason as in Weinberg's discussion [44] of the Thomas model when the general decomposition of the energy momentum depends on the definition of the temperature. For a complete discussion of the heat flow we would need x -dependent χ and β in eq.(115). If there is no heat flow ($\sigma_1 = \delta_0$) then in the fluid rest frame we obtain the Navier-Stokes term Δu^j in the fluid equation (139). We cannot formulate a closed hydrodynamic scheme in the sense of Israel and Stewart [45] in the approximation which we are applying in this section, in order to check its causality and thermodynamic stability [46]. For this purpose we would have to discuss the hydrodynamic evolution of the general state (115) to higher orders of perturbation.

10 Summary

The assumption of a continuous mass spectrum enables us to work in an explicitly covariant way treating all components of the momentum on the same footing. We have studied the possible relativistic diffusions either by searching positive definite diffusion matrices or approximating random dynamics by a diffusion. The latter method shows that the relativistic diffusion really arises in physical systems. We have studied in more detail the random electromagnetic field applied in quantum optics for a description of photons in cavities at finite temperature. We identified the non-zero expectation value of the electric current as a possible source of friction leading to the Jüttner equilibrium. The resulting diffusion may have an application to relativistic streams of particles encountered in astrophysics [47] and in high-energy physics [48]. In secs.7-9 we have shown how to apply standard methods (widely used in the case of the Boltzmann equation) for a statistical and hydrodynamic description of a stream of relativistic diffusing particles satisfying the diffusion equation. In this way we could relate methods based on the diffusion equation [48] to the ones using the hydrodynamic equations in heavy ion physics [37][49]. The relativistic diffusion equation can also supply new ways of approaching the problems of relativistic statistical physics and relativistic hydrodynamics by means of methods applying differential equations rather than the integral equations of the Boltzmann type.

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